Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 5, 1997

BEHAVIOR OF A GAS BUBBLE IN A VISCOUS OSCILLATING LIQUID IN THE PRESENCE OF GRAVITY

V. L. Sennitskii

UDC 532.529

The phenomenon of the predominantly unidirectional motion of a gas bubble in a vibrating liquid has previously been discovered theoretically and demonstrated experimentally by the author [1, 2]. The essence of this phenomenon is as follows. There is a closed vessel filled with a liquid containing a gas bubble. Gravity is absent. The vessel accomplishes specified periodic oscillations and is deformed in a specified manner. As a consequence, the gas bubble moves in a given direction (in the positive or negative direction of the axis along which the vessel oscillates, depending on the oscillations and deformations of the vessel).

The existence of this phenomenon, which is interesting in itself, suggests, in particular, that the behavior of the gas bubble can be unusual in the presence of gravity because of the oscillations and deformations of the vessel.

A similar effect [3] can also occur for which the following factors are critical: the presence of gravity and the condition that the liquid with a gas bubble is placed in an open vessel or a similar liquid fills a closed vessel only partially. When the vessel oscillates vertically, the bubble neither rises nor sinks at a specific depth, rises slower at a smaller depth, or sinks at a larger depth.

In the present paper, the problem of the motion of a gas bubble in a viscous incompressible oscillating liquid in the presence of gravity is considered. As in [1], a bubble-containing liquid fills a closed vessel which oscillates and is subjected to deformation. Unlike as in [1], the solution for not too small oscillations and deformations of the vessel is obtained, but the Reynolds number is assumed to be small. In particular, it is shown that the gas bubble can either rise faster or slower, neither rise nor sink, or sink under the action of the oscillations and deformations of the vessel. An important circumstance is that the realization of all these types of motion of the gas bubble is not restricted by the condition of bubble location at a definite depth.

1. A bubble is in a viscous incompressible liquid which is bounded from outside by the surface of a closed vessel formed by deformable solid walls and by absolutely solid walls which are rigidly connected to each other. The vessel accomplishes the prescribed periodic (with period T) translational oscillations along the z axis relative to an inertial rectangular coordinate system X, Y, Z. Simultaneously, the vessel is deformed in a specified manner (it compresses and expands). There is a constant gravity [the acceleration of gravity $\mathbf{g} = (0, 0, -g)$, $g \geqslant 0$]. The location of the bubble relative to the coordinate system X, Y, Z is characterized by the radius vector

$$\mathbf{S} = \frac{1}{Q} \iiint\limits_{\Omega_{XYZ}} \mathbf{R}\, dX dY dZ,$$

where $\mathbf{R} = (X, Y, Z)$, Ω_{XYZ} is the region occupied by the gas, and Q is the gas volume ($\mathbf{S}$ is the radius vector of the center of inertia of the bubble). The liquid flow is considered relative to the rectangular coordinate system $X_1 = X - S_X$, $X_2 = Y - S_Y$, and $X_3 = Z - S_Z$ (S_X, S_Y, and S_Z are, respectively, the X, Y, and Z components of the vector $\mathbf{S}$). The smallest distance from the bubble to the vessel walls is large compared with the largest size of the bubble; therefore, the vessel walls are assumed to be infinitely far from the bubble.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 5, pp. 73–79, September–October, 1997. Original article submitted March 1, 1996.

The velocity $\mathbf{V}$ of the liquid flow is subject to the condition

$$\mathbf{V} \sim \tilde{U}\mathbf{k} - \frac{d\mathbf{S}}{dt} \quad \text{for} \quad X_1^2 + X_2^2 + X_3^2 \to \infty,$$

where t is time, $\tilde{U} = \text{Real} \sum_{m=1}^{\infty} U_m e^{2m\pi i t/T}$ (U_m are constants), and $\mathbf{k} = (0,0,1)$ ($\tilde{U}\mathbf{k}$ is the velocity of the oscillatory motion of the vessel relative to the coordinate system X, Y, Z). Accordingly, the pressure P in the liquid is subject to the condition

$$P \sim -\rho\Big(\frac{d\tilde{U}}{dt} + g\Big)X_3 + \tilde{P} \quad \text{for} \quad X_1^2 + X_2^2 + X_3^2 \to \infty,$$

where ρ is the density of the liquid and $\tilde{P}$ is a function of t. The dependence of $\tilde{P}$ on t is determined by how the vessel is deformed. It is assumed that

$$\tilde{P} = P_0 + \text{Real} \sum_{m=1}^{\infty} P_m e^{2m\pi i t/T},$$

where P_0 ($P_0 \geqslant 0$) and P_m are constants. The flow of the liquid is not dependent on the initial conditions. In the absence of gravity and oscillations and deformations of the vessel [for $g = 0$ and $U_m = P_m = 0$ ($m = 1, 2, \ldots$)], the bubble is a ball $\sqrt{X_1^2 + X_2^2 + X_3^2} \leqslant A_0$ (A_0 is a constant), $\mathbf{V} = 0$, and $P = P_0$. The pressure P_g and volume of the gas are related by the adiabatic equation

$$P_g Q^\gamma = P_{g0} Q_0^\gamma,$$

where γ is the adiabatic exponent, $P_{g0} = P_0 + 2\sigma/A_0$ (σ is the coefficient of surface tension), and $Q_0 = (4\pi/3)A_0^3$. It is necessary to reveal the motion of the bubble relative to the coordinate system X, Y, Z, i.e., to find the dependence of $\mathbf{S}$ on t.

Assume that $\tau = t/T$, $x_1 = X_1/A_0$, $x_2 = X_2/A_0$, $x_3 = X_3/A_0$, $\mathbf{r} = (x_1, x_2, x_3)$, $r = |\mathbf{r}|$, Γ is the surface bounding the region $\Omega_{x_1x_2x_3}$ occupied by the gas (the free boundary of the region occupied by the liquid), H is the average curvature of Γ, $\eta = A_0H - 1$, $\mathbf{n}$ is a unit external normal to Γ, Ξ is the velocity of Γ in the direction of $\mathbf{n}$, $\xi = T\Xi/A_0$, $\mathbf{v} = (v_i) = T\mathbf{V}/A_0$, $p = T^2(P - P_0)/(\rho A_0^2)$, $\mathbf{w} = (1/A_0)d\mathbf{S}/d\tau$, ν is the kinematic coefficient of liquid viscosity, $\text{Re} = A_0^2/(\nu T)$ is the Reynolds number, $\mathbf{P}$ is the stress tensor in the liquid, $\mathbf{I} = (I_{ij})$ is the unit tensor, $\mathbf{p} = (p_{ij}) = T^2(\mathbf{P} + P_0\mathbf{I})/(\rho A_0^2)$ [$p_{ij} = -pI_{ij} + (1/\text{Re})(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$], $\hat{P}$ is the largest value of $|\tilde{P} - P_0|$, $\tilde{p} = (\tilde{P} - P_0)/\hat{P} = \text{Real} \sum_{m=1}^{\infty} p_m e^{2m\pi i\tau}$, $\hat{U}$ is the largest value of $|\tilde{U}|$, $\tilde{u} = \tilde{U}/\hat{U} = \text{Real} \sum_{m=1}^{\infty} u_m e^{2m\pi i\tau}$, $\boldsymbol{\alpha} = (0,0,\alpha) = -T^2\mathbf{g}/A_0$, $\varepsilon = \hat{U}T/A_0$, $K = \hat{P}T/(\rho\nu)$, $\lambda = \sigma T^2/(\rho A_0^3)$, $\mu = P_{g0}T^2/(\rho A_0^2)$, and $p_g = T^2(P_g - P_{g0})/(\rho A_0^2) = \mu(Q_0^\gamma/Q^\gamma - 1)$.

The equation of the surface Γ, the Navier–Stokes and continuity equations, and the conditions that must be satisfied on Γ for $r \to \infty$ have the following form:

$$\chi = 0 \tag{1.1}$$

$$(\chi < 0 \ \text{in} \ \Omega_{x_1x_2x_3}, \qquad \chi > 0 \ \text{outside of} \ \Omega_{x_1x_2x_3});$$

$$\frac{\partial \mathbf{v}}{\partial \tau} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p - \frac{1}{\text{Re}}\Delta\mathbf{v} + \frac{d\mathbf{w}}{d\tau} + \boldsymbol{\alpha} = 0; \tag{1.2}$$

$$\nabla \cdot \mathbf{v} = 0; \tag{1.3}$$

$$\mathbf{n} \cdot \mathbf{v} - \xi = 0, \qquad \mathbf{n} \cdot \mathbf{p} + (p_g - 2\lambda\eta)\mathbf{n} = 0 \quad \text{on} \ \Gamma; \tag{1.4}$$

$$\mathbf{v} \sim \varepsilon\tilde{u}\mathbf{k} - \mathbf{w}, \quad p \sim -\Big(\varepsilon\frac{d\tilde{u}}{d\tau} + \alpha\Big)x_3 + \frac{K}{\text{Re}}\tilde{p} \quad \text{for} \ r \to \infty. \tag{1.5}$$

The following relation also must be satisfied:

$$\iiint\limits_{\Omega_{x_1x_2x_3}} \mathbf{r}\,dx_1dx_2dx_3 = 0 \tag{1.6}$$

(the center of inertia of the bubble coincides with the origin of coordinates x_1, x_2, x_3).

2. Problem (1.1)–(1.6) for $\alpha = 0$ ($\mathbf{g} = 0$) was formulated and solved approximately in [1]; the expansions of χ, $\mathbf{v}$, p, and $\mathbf{w}$ for $\text{æ} = (K/\text{Re}) \to 0$, $\varepsilon \to 0$, and constant x_1, x_2, x_3, τ, Re, λ, and μ were considered in that paper as well. Another approach to the study of problem (1.1)–(1.6) is given below, where the expansions of χ, $\mathbf{v}$, p, and $\mathbf{w}$ for $\text{Re} \to 0$ and constant x_1, x_2, x_3, τ, α ε, K, λ, and μ are considered.

Let us assume that, for $\text{Re} \to 0$,

$$\chi \sim \chi_{(0)} + \text{Re}\,\chi_{(1)}, \quad \mathbf{v} \sim \mathbf{v}_{(0)} + \text{Re}\,\mathbf{v}_{(1)}, \quad p \sim \frac{1}{\text{Re}}\,p_{(0)} + p_{(1)}, \quad \mathbf{w} \sim \mathbf{w}_{(0)} + \text{Re}\,\mathbf{w}_{(1)}. \tag{2.1}$$

In accordance with (1.1)–(1.6) and (2.1), in the Lth ($L = 0$ and 1) approximation we have

$$\chi_{(0)} + L\text{Re}\chi_{(1)} = 0, \tag{2.2}$$

which is the equation of the surface $\Gamma_{(L)}$ bounding the region $\Omega_{(L)}$ occupied by the gas;

$$\nabla p_{(L)} - \Delta \mathbf{v}_{(L)} = -L\Big[\frac{\partial \mathbf{v}_{(0)}}{\partial \tau} + (\mathbf{v}_{(0)}\cdot\nabla)\mathbf{v}_{(0)} + \frac{d\mathbf{w}_{(0)}}{d\tau} + \alpha\Big]; \tag{2.3}$$

$$\nabla\cdot\mathbf{v}_{(L)} = 0; \tag{2.4}$$

$$\begin{gathered}\lim_{\text{Re}\to 0}\Big[\text{Re}^{-L}(\mathbf{n}_{(L)}\cdot\mathbf{v} - \xi_{(L)})\Big|_{\Gamma_{(L)}}\Big] = 0,\\ \lim_{\text{Re}\to 0}\Big\{\text{Re}^{1-L}[\mathbf{n}_{(L)}\cdot\mathbf{p} + (p_{g(L)} - 2\lambda\eta_{(L)})\,\mathbf{n}_{(L)}]\Big|_{\Gamma_{(L)}}\Big\} = 0;\end{gathered} \tag{2.5}$$

$$\begin{gathered}\mathbf{v}_{(L)} \sim (1-L)(\varepsilon\tilde{u}\mathbf{k} - \mathbf{w}_{(0)}) - L\mathbf{w}_{(1)},\\ p_{(L)} \sim (1-L)\,K\tilde{p} - L\Big(\varepsilon\frac{d\tilde{u}}{d\tau} + \alpha\Big)x_3 \quad \text{for } r\to\infty;\end{gathered} \tag{2.6}$$

$$\iiint\limits_{\Omega_{(L)}} \mathbf{r}\,dx_1dx_2dx_3 = 0, \tag{2.7}$$

where $\mathbf{n}_{(L)}$, $\eta_{(L)}$, $\xi_{(L)}$, and $p_{g(L)}$ are, respectively, $\mathbf{n}$, η, ξ, and p_g for $\Gamma = \Gamma_{(L)}$.

Let $L = 0$. For the zero approximation, the bubble volume changes under the action of deformations of the vessel; the bubble, however, cannot move relative to the liquid being at infinity because of the vessel's oscillations and the influence of gravity. The bubble is a ball $r \leqslant 1 + a_{(0)}$ whose center moves with velocity $\tilde{U}\mathbf{k}$ relative to the coordinates X, Y, Z. The liquid flow is symmetrical with respect to the origin of coordinates x_1, x_2, x_3. Relation (2.7) is satisfied. Problem (2.2)–(2.7) has the solution

$$\chi_{(0)} = r - 1 - a_{(0)}; \tag{2.8}$$

$$\mathbf{v}_{(0)} = (1 + a_{(0)})^2\,\frac{da_{(0)}}{d\tau}\,\frac{\mathbf{r}}{r^3}; \tag{2.9}$$

$$p_{(0)} = K\tilde{p}; \tag{2.10}$$

$$\mathbf{w}_{(0)} = w_{(0)}\mathbf{k}. \tag{2.11}$$

Here $w_{(0)} = \varepsilon\tilde{u}$ and

$$a_{(0)} = -1 + C\exp\Big(-\frac{1}{4}K\int\limits_{\tau_*}^{\tau}\tilde{p}\,d\tau\Big), \tag{2.12}$$

where τ_* is a constant and $C = 1 + a_{(0)}\big|_{\tau=\tau_*}$ $(C > 0)$. The constant C is not determined in the consideration of the zero approximation.

Let $L = 1$. Let us assume that in the first approximation, the bubble is a ball $r \leqslant 1 + a_{(0)} + \operatorname{Re} a_{(1)}$. Relation (2.7) is then satisfied. Conditions (2.5) and (2.6) are reduced to

$$v_{(1)r} - 2(1 + a_{(0)})^2 \frac{da_{(0)}}{d\tau} \frac{a_{(1)}}{r^3} - \frac{da_{(1)}}{d\tau} = 0,$$

$$-p_{(1)} + 2\frac{\partial v_{(1)r}}{\partial r} + a_{(1)} \frac{\partial}{\partial r}\Big(-p_{(0)} + 2\frac{\partial v_{(0)r}}{\partial r}\Big) + \mu[(1 + a_{(0)})^{-3\gamma} - 1] + 2\lambda \frac{a_{(0)}}{1 + a_{(0)}} = 0, \tag{2.13}$$

$$\frac{1}{r}\frac{\partial v_{(1)r}}{\partial \theta} + \frac{\partial v_{(1)\theta}}{\partial r} - \frac{v_{(1)\theta}}{r} = 0, \quad \frac{\partial v_{(1)\varphi}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_{(1)r}}{\partial \varphi} - \frac{v_{(1)\varphi}}{r} = 0 \quad \text{for } r = 1 + a_{(0)},$$

where θ is the angle between the vectors $(0,0,1)$ and (x_1, x_2, x_3) $(0 \leqslant \theta \leqslant \pi)$, φ is the angle between the vectors $(1,0,0)$ and $(x_1, x_2, 0)$ $(0 \leqslant \varphi < 2\pi)$ (r, θ, and φ are the spherical coordinates); $v_{(1)r}$, $v_{(1)\theta}$, and $v_{(1)\varphi}$ are the r, θ, and φ components of the vector $\mathbf{v}_{(1)}$, respectively. Problem (2.3), (2.4), (2.6), and (2.13) has the solution

$$v_{(1)r} = \frac{d[(1 + a_{(0)})^2 a_{(1)}]/d\tau}{r^2} + w_{(1)}\Big(\frac{1 + a_{(0)}}{r} - 1\Big)\cos\theta,$$

$$v_{(1)\theta} = w_{(1)}\Big(1 - \frac{1 + a_{(0)}}{2r}\Big)\sin\theta, \qquad v_{(1)\varphi} = 0, \tag{2.14}$$

$$p_{(1)} = w_{(1)}(1 + a_{(0)})\frac{\cos\theta}{r^2} - \Big(\varepsilon\frac{d\tilde{u}}{d\tau} + \alpha\Big) r\cos\theta + r\frac{\partial v_{(0)r}}{\partial r} - \frac{1}{2}v_{(0)r}^2;$$

$$\mathbf{w}_{(1)} = w_{(1)}\mathbf{k}. \tag{2.15}$$

Here

$$w_{(1)} = \frac{1}{3}\Big(\varepsilon\frac{d\tilde{u}}{d\tau} + \alpha\Big)(1 + a_{(0)})^2;$$

$$a_{(1)} = (1 + a_{(0)})\Big\{\Big[\frac{a_{(1)}}{1 + a_{(0)}} + \frac{1}{4}(1 + a_{(0)})\frac{da_{(0)}}{d\tau}\Big]\Big|_{\tau=\tau_{**}} - \frac{1}{4}(1 + a_{(0)})\frac{da_{(0)}}{d\tau} + \int_{\tau_{**}}^{\tau} f\,d\tau\Big\}$$

$$\Big[\tau_{**} \text{ is a constant}, \ f = \frac{1}{4}\Big\{-\frac{1}{2}\Big(\frac{da_{(0)}}{d\tau}\Big)^2 + \mu[(1 + a_{(0)})^{-3\gamma} - 1] + 2\lambda\frac{a_{(0)}}{1 + a_{(0)}}\Big\}\Big].$$

Thus, $\chi_{(1)} = -a_{(1)}$, and the solution (2.14) and (2.15) of problem (2.3), (2.4), (2.6), and (2.13) is a solution of problem (2.2)–(2.7).

According to (2.12), the quantity f is a periodic (with period 1) function of τ. The quantity $a_{(1)}$ must be a limited function of τ. This condition is satisfied if and only if the following relation is satisfied:

$$\int_{\tau}^{\tau+1} f\,d\tau = 0. \tag{2.16}$$

According to (2.12) and (2.16), we have

$$c_1 C^{3\gamma+2} + c_2 C^{3\gamma} + c_3 C^{3\gamma-1} = c_4, \tag{2.17}$$

where

$$c_1 = \int_{\tau}^{\tau+1}\Big(\frac{dh}{d\tau}\Big)^2 d\tau; \ c_2 = \mu - 2\lambda; \ c_3 = 2\lambda\int_{\tau}^{\tau+1} h^{-1}d\tau; \ c_4 = \mu\int_{\tau}^{\tau+1} h^{-3\gamma}\,d\tau$$

$$\Big[h = \exp\Big(-\frac{1}{4}K\int_{\tau_*}^{\tau}\tilde{p}\,d\tau\Big)\Big].$$

The left-hand side of (2.17) is equal to zero at $C = 0$; for $C > 0$, it is positive and infinitely increases monotonically with increasing C; the right-hand side of (2.17) is positive and does not vary with C. As a consequence, there is a positive value of C which satisfies relation (2.17), and this value of C is unique. Thus, the constant C is determined in the consideration of the first approximation.

3. With the use of (2.11) and (2.15), we obtain

$$\mathbf{S} = A_0 \int_0^{\tau} (w_{(0)} + \mathrm{Re}\, w_{(1)})\, d\tau\ \mathbf{k} + \mathbf{s}_0, \tag{3.1}$$

where $\mathbf{s}_0$ is a constant. Relation (3.1) determines approximately the dependence of $\mathbf{S}$ on t.

In particular, it follows from (3.1) that the bubble moves along the Z axis, and its motion consists of oscillations and the translation of the constant velocity

$$\overline{\mathbf{W}} = \overline{W}\mathbf{k}, \tag{3.2}$$

where

$$\overline{W} = \frac{A_0 \mathrm{Re}}{3T} \int_{\tau}^{\tau+1} \left(\varepsilon \frac{d\tilde{u}}{d\tau} + \alpha\right)(1 + a_{(0)})^2\, d\tau.$$

In accordance with (3.2), if $\mathbf{g} \neq 0$, the bubble rises for

$$\overline{W} > 0,$$

sinks for

$$\overline{W} < 0,$$

neither rises nor sinks for

$$\overline{W} = 0,$$

rises faster than in the absence of oscillations and deformations of the vessel for

$$\int_{\tau}^{\tau+1} \left(\varepsilon \frac{d\tilde{u}}{d\tau} + \alpha\right)(1 + a_{(0)})^2\, d\tau > \alpha,$$

and rises slower than in the absence of oscillations and deformations of the vessel for

$$0 < \int_{\tau}^{\tau+1} \left(\varepsilon \frac{d\tilde{u}}{d\tau} + \alpha\right)(1 + a_{(0)})^2\, d\tau < \alpha.$$

4. Let us compare Eq. (3.2) obtained above for the velocity of bubble translation with that obtained in [1].

Let $\mathbf{g} = 0$ and

$$\langle \mathbf{w} \rangle = \int_{\tau}^{\tau+1} \mathbf{w}\, d\tau. \tag{4.1}$$

According to (3.1), (3.2), and (4.1), we have

$$\langle \mathbf{w} \rangle \sim \mathrm{Re} \langle \mathbf{w} \rangle_{\mathrm{Re}} \quad \text{for } \mathrm{Re} \to 0 \ \text{(and constant } K,\ \varepsilon,\ \lambda, \text{ and } \mu), \tag{4.2}$$

where

$$\langle \mathbf{w} \rangle_{\mathrm{Re}} = \frac{1}{3} \varepsilon \int_{\tau}^{\tau+1} \frac{d\tilde{u}}{d\tau}(1 + a_{(0)})^2\, d\tau\ \mathbf{k} = \frac{T}{A_0 \mathrm{Re}} \overline{\mathbf{W}}. \tag{4.3}$$

From (4.3), it follows that

$$\langle \mathbf{w} \rangle_{\mathrm{Re}} \sim K \langle \mathbf{w} \rangle_{K,\mathrm{Re}} \text{ at } K \to 0 \text{ (and constant Re, } \varepsilon, \lambda, \text{ and } \mu), \quad (4.4)$$

where

$$\langle \mathbf{w} \rangle_{K,\mathrm{Re}} = \frac{1}{12} \varepsilon \,\mathrm{Real} \sum_{m=1}^{\infty} p_m^* u_m \mathbf{k} \quad (4.5)$$

(p_m^* are the constants that are complex-conjugated with p_m). It is obvious from (4.5) that

$$\langle \mathbf{w} \rangle_{K,\mathrm{Re}} \sim \varepsilon \langle \mathbf{w} \rangle_{\varepsilon,K,\mathrm{Re}} \text{ at } \varepsilon \to 0 \text{ (and constant Re, } K, \lambda, \mu), \quad (4.6)$$

where

$$\langle \mathbf{w} \rangle_{\varepsilon,K,\mathrm{Re}} = \frac{1}{12} \mathrm{Real} \sum_{m=1}^{\infty} p_m^* u_m \mathbf{k}.$$

According to (4.2), (4.4), and (4.6), the relation $\langle \mathbf{w} \rangle \sim \mathrm{Re} K \varepsilon \langle \mathbf{w} \rangle_{\varepsilon,K,\mathrm{Re}}$ is satisfied for $\varepsilon \to 0$ (and constant Re, K, λ, and μ), $K \to 0$ (and constant Re, ε, λ, and μ), $\mathrm{Re} \to 0$ (and constant K, ε, λ, and μ). In [1], the expression $(A_0/T)\varepsilon æ \bar{w} \mathbf{k}$ for the velocity of bubble translation was obtained. According to [1], the quantity $\varepsilon æ \bar{w} \mathbf{k}$ is the main term $\varepsilon æ \langle \mathbf{w} \rangle_{æ,\varepsilon}$ of the expansion of $\langle \mathbf{w} \rangle$ for $æ = (K/\mathrm{Re}) \to 0$ (and constant Re, ε, λ, and μ) and for $\varepsilon \to 0$ (and constant Re, $æ$, λ, and μ). It is easy to see that the main term of the expansion of $\varepsilon æ \langle \mathbf{w} \rangle_{æ,\varepsilon}$ for $\mathrm{Re} \to 0$ (and constant K, ε, λ, and μ) coincides with $\mathrm{Re}\, K \varepsilon \langle \mathbf{w} \rangle_{\varepsilon,K,\mathrm{Re}}$. This gives a basis for considering that the compared expressions for the velocity of bubble translation are in accordance with each other.

5. The investigation presented allows one, in particular, to draw the following conclusion. If oscillations and deformations of the vessel are such that the bubble neither rises nor sinks, then the bubble can move in any prescribed direction owing to additional oscillations of the vessel (along the properly oriented axis). Thus, the phenomenon of predominantly unidirectional motion of a gas bubble in a vibrating liquid can be understood in a wider sense that the bubble moves in any prescribed direction under the action of oscillations and deformations of the vessel both in the presence and in the absence of gravity.

REFERENCES

1. V. L. Sennitskii, "On the motion of a gas bubble in a viscous vibrating liquid," *Prikl. Mekh. Tekh. Fiz.*, No. 6, 107–113 (1988).
2. V. L. Sennitskii, "Predominantly unidirectional motion of a gas bubble in a vibrating liquid," *Dokl. Akad. Nauk SSSR*, **319**, No. 1, 117–119 (1991).
3. H. H. Bleich, "Effect of vibrations on the motion of small gas bubbles in a liquid," *Jet Propulsion*, **26**, No. 11, 958–964, 978 (1956).